B.Sc. Part–II (Semester–III) Examination

MATHEMATICS

(Elementary Number Theory)

Paper—VI

Time : Three Hours] [Maximum Marks : 60

N.B. :— (1) Q. No. **1** is compulsory ; attempt it only once.

(2) Attempt **ONE** question from each unit.

1. Choose correct alternatives :—

(1) If $a \mid bc$ and $(a, b) = 1$ then

(a) $a \mid b$ (b) $a \mid c$

(c) $c \mid a$ (d) $b \mid a$

(2) A necessary and sufficient condition for $[a, b] = ab$ is :

(a) $[a, b] = 1$ (b) $ab = 1$

(c) $(a, b) = 1$ (d) None of these

(3) The conjecture 'Every odd integer is the sum of at most three primes' is given by :

(a) Euler (b) Goldbach

(c) Eratosthenes (d) None of these

(4) If p_n is the n^{th} prime number then

(a) $p_n \leq 2^{2^n}$ (b) $p_n \leq 2^{n-1}$

(c) $p_n \leq 2^{2^{n-1}}$ (d) $p_n \leq 2$

(5) The set {0, 1, 2, 3} is complete system of residue modulo :

(a) 3 (b) 4

(c) 5 (d) 2

(6) The function f is multiplicative if

(a) $f(mn) = f(m) + f(n)$ (b) $f(mn) = f(m) . f(n)$

(c) $f(mn) = f(m) - f(n)$ (d) None of these

(7) The statement $a \equiv b \pmod{m}$ is equivalent to

(a) $b \equiv a \pmod{m}$ (b) $(a - b) \equiv 0 \pmod{m}$

(c) Both (a) and (b) are true (d) Both (a) and (b) are false

(8) If n = 18 then the value of $\tau(18)$ and $\sigma(18)$ are :

(a) 6 and 39 (b) 6 and 40

(c) 39 and 40 (d) 6 and 7

(9) If P is prime divisor of Fermat number $F_n = 2^{2^n} + 1$ then $O_p(2) =$ ______.

(a) 2^n (b) 2^{n-1}

(c) 2^{n+1} (d) 2^{2^n}

(10) The order of 2 modulo 7 is :

(a) 3 (b) 2

(c) 7 (d) 1 10×1=10

UNIT—I

2. (a) Let a and b be integers, not both zero. Then prove that there exist integers x and y such that (a, b) = xa + yb. 5

(b) If $a, b \in I$, $b \neq 0$ and $a = bq + r$, $0 \leq r < b$ then prove that (a, b) = (b, r). 3

(c) Define :

(i) Relatively prime

(ii) Greatest Common Divisor 2

3. (p) Let a, b, c be positive integers. Then prove that

$$[a, b, c] = \frac{abc}{(ab, bc, ca)}.$$ 3

(q) If (a, b) = 1 then prove that (ac, b) = (c, b). 3

(r) Find the gcd of 275 and –200 and express it in the form xa + yb. 4

UNIT—II

4. (a) If m and n are distinct non-negative integers then prove that $(F_m, F_n) = 1$. 5

(b) Prove that there are infinitely many number primes of the form 4n + 3, where n is a positive integer. 5

5. (p) Prove that the Fermat number F5 is divisible by 641 and hence composite. 5

(q) Prove that every positive integer greater than one has at least one prime divisor. 5

UNIT—III

6. (a) Let $a_1, a_2, b_1, b_2 \in I$ such that $a_1 \equiv b_1 \pmod{m}$ and $a_2 \equiv b_2 \pmod{m}$ then prove that :

(i) $(a_1 \pm a_2) \equiv (b_1 \pm b_2) \pmod{m}$

(ii) $a_1a_2 \equiv b_1b_2 \pmod{m}$ 4

(b) If $a \equiv b \pmod{m}$ then prove that $a^n \equiv b^n \pmod{m}$. 3

(c) Solve the congruence using inverse of a modulo m, $3x \equiv 1 \pmod{125}$. 3

7. (p) Solve the system of three congruences $x \equiv 1 \pmod{4}$, $x \equiv 0 \pmod{3}$, $x \equiv 5 \pmod{7}$. 4

(q) Prove that $ca \equiv cb \pmod{m} \Leftrightarrow a \equiv b \left(\bmod \dfrac{m}{d}\right)$, where $d = (c, m)$. 3

(r) Find the remainder of 43^{289} is divided by 7. 3

UNIT—IV

8. (a) Let $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots\ldots p_m^{a_m}$ be the prime factorisation of the position integer n. Then prove that

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_m}\right). \quad 3$$

(b) If F is multiplicative function and $F(n) = \sum_{d|n} f(d)$, then prove that f is also multiplicative. 3

(c) Solve the linear congruence $3x \equiv 5 \pmod{16}$ by using Euler's theorem. 4

9. (p) Show that the sum of $\phi(n)$ positive integers less than n (> 1) and relatively prime to n is $\dfrac{n}{2}\phi(n)$. 3

(q) Let the positive integer n have prime factorisation

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots\ldots p_m^{a_m}$$

Then prove that $\tau(n) = (a_1 + 1)(a_2 + 1) \ldots\ldots (a_m + 1) = \prod_{i=1}^{m}(a_i + 1)$ and

$$\sigma(n) = \frac{p_1^{a_1+1}-1}{p_1-1} \cdot \frac{p_2^{a_2+1}-1}{p_2-1} \ldots\ldots \frac{p_m^{a_m+1}-1}{p_m-1} = \prod_{i=1}^{m} \frac{p_i^{a_i+1}-1}{p_i-1}. \quad 3$$

(r) For each positive integer $n \geq 1$, prove

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & , \quad n = 1 \\ 0 & , \quad n > 1 \end{cases}.$$ 4

UNIT—V

10. (a) Let p be prime number and $d|(p-1)$. Then prove that the congruence $x^d - 1 \equiv 0 \pmod{p}$ has exactly d solutions. 5

(b) Find all primitive roots of $p = 17$. 5

11. (p) If $O_m(a) = n$ then prove that $O_m(a^k) = \frac{n}{(n, k)}$ where k is a positive integer. 5

(q) If a and m are relatively prime positive integers and if a is primitive root of m then prove that the integers $a, a^2, \ldots\ldots, a^{\phi(m)}$ form a reduced residue set modulo m. 5